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Conformal invariance in the Leigh-Strassler deformed $N = 4$ SYM theory

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Abstract: We consider a full Leigh-Strassler deformation of the $\mathcal{N} = 4$ SYM theory and look for conditions under which the theory would be conformally invariant and finite. Applying the algorithm of perturbative adjustments of the couplings we construct a family of theories which are conformal up to 3 loops in the non-planar case and up to 4 loops in the planar one. We found particular solutions in the planar case when the conformal condition seems to be exhausted in the one loop order. Some of them happen to be unitary equivalent to the real beta-deformed $\mathcal{N} = 4$ SYM theory, while others are genuine. We present the arguments that these solutions might be valid in any loop order.

Keywords: Renormalization Group, AdS-CFT Correspondence, Supersymmetric gauge theory.

Contents

1. Introduction

During the last decade much attention has been paid to the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) and its deformations obtained by the orbifold [1] or orientifold [2] projection, or by adding the marginal deformations [3] to the Lagrangian. Such deformations lead to a theory with less supersymmetry but inheriting some attractive features of the original $\mathcal{N} = 4$ SYM theory, namely, the conformal invariance, integrability [4, 5] in the planar limit, and, especially, its connection with the dual string theory via the AdS/CFT correspondence. This way it becomes possible to investigate nonperturbative features of these theories.

Since the original version of the AdS/CFT correspondence [6] there have appeared a lot of its modifications [7]. At the present time, it is not clear how to build gravity dual to an arbitrary gauge theory or which properties of the gauge theories are necessary for existence of this correspondence. However it is obvious that conformal invariance [8] of the gauge theory plays a significant role in this matter. As it was already mentioned, the Leigh-Strassler deformation of the $\mathcal{N} = 4$ SYM theory [3] breaks the initial supersymmetry to $\mathcal{N} = 1$ supersymmetry and the $\mathrm{SU}(4)_R$ symmetry down to $\mathrm{U}(1)_R$. One of such examples is the so-called β-deformation of the original $\mathcal{N} = 4$ SYM theory. Its gravity dual was constructed by Lunin and Maldacena [9] and a significant role in this duality is played by the $\mathrm{U}(1) \times \mathrm{U}(1)$ global symmetry of the β-deformed theory which was associated with isometries of the deformed $AdS_5 \times \tilde{S}_5$ background. There are also attempts to construct the gravity dual to the full Leigh-Strassler deformation [10 – 12].

From the field theory side the investigation of the β-deformation of the $\mathcal{N} = 4$ SYM theory was dedicated mainly to finding the conditions of conformal invariance [13 – 16] and finiteness [15] of the theory, and to investigation of Chiral Primary Operators(CPO) [16, 17]. In the real β case [14], it was shown that the theory is exactly conformal in the planar limit. For general β the condition of conformal invariance = finiteness in the planar limit was found up to four loops in [18]. In the nonplanar case, the conformal condition was found up to three loops in [16] and recently the first step towards the four-loop answer was made in [19].

The case of the full Leigh-Strassler deformation was less investigated from the quantum field theory side. The one-loop conformal condition was obtained almost five years ago [20, 21] while the three-loop anomalous dimension was recently calculated in [22] using the results of the papers [23, 24]. Their result, however, seems not to coincide with us and with the β-deformed case from [16]. Also, some CPO were investigated in [17]. In this paper, we look for the conformal invariance of the full Leigh-Strassler deformation. Using the dimensional regularization(reduction) we found conditions of conformal invariance up to four loops in the planar limit and up to three loops in the non-planar one.

There are special cases when the conformal conditions are exhausted in the one-loop order. In case of the β-deformed theory in the planar limit, this corresponds to real values of β. We also found such solutions for the full Leigh-Strassler deformation. However, some of these solutions happen to be unitary equivalent to the β-deformed case. This gives us a useful cross check of our calculations. At the same time, also in the planar limit, there exist non-trivial solutions which are not reduced to the β-deformed ones. We present them below and conjecture that they might be valid in any loop order.

This family of solutions does not possess any global symmetries, except for Z_3, and has connections with the β-deformed $\mathcal{N} = 4$ SYM at particular points. It would be very interesting to understand their origin from the string theory side and build the corresponding dual gravity background.

2. The Leigh-Strassler deformation of the $\mathcal{N} = 4$ SYM theory

The so-called Leigh-Strassler deformation can be obtained by modification of the superpotential in the original $\mathcal{N} = 4$ SYM theory written in terms of $\mathcal{N} = 1$ superfields:

$$S = \int d^8z Tr\left(e^{-gV}\bar{\Phi}_i e^{gV}\Phi^i\right) + \left(\frac{1}{2g^2}\int d^6z Tr(W^\alpha W_\alpha) + \int d^6z\, \mathcal{W} + h.c.\right) \tag{2.1}$$

in such a way

$$\begin{aligned} \mathcal{W}_{N=4\ SYM} &= ig(Tr(\Phi_1\Phi_2\Phi_3) - Tr(\Phi_1\Phi_3\Phi_2)) \rightarrow \\ \mathcal{W}_{LS\ SYM} &= i\left[h_1 Tr(\Phi_1\Phi_2\Phi_3) - h_2 Tr(\Phi_1\Phi_3\Phi_2) + \frac{h_3}{3}\sum_{i=1}^{3} Tr(\Phi_i^3)\right], \end{aligned} \tag{2.2}$$

where Φ_i with $i = 1, 2, 3$ are the three chiral superfields of the original $\mathcal{N} = 4$ SYM theory in the adjoint representation of the gauge group SU(N), and the couplings h_1, h_2, h_3 are in general complex. The β-deformed case in the same notation corresponds to

$$h_1 = hq,\ \ h_2 = h/q,\ \ q = e^{i\pi\beta} \text{ and } h_3 = 0.$$

The Leigh-Strassler deformed superpotential breaks the $\mathrm{SU}(4)_R$ symmetry of the original $\mathcal{N}=4$ theory down to $\mathrm{U}(1)_R$. In addition, it is invariant under cyclic permutations of $(\Phi_1,\ \Phi_2,\ \Phi_3)$ and exchange: $\beta \leftrightarrow 1-\beta$ or in our notation $h_1 \leftrightarrow -h_2$.

In case of interest, as in any $\mathcal{N}=1$ SYM theory formulated in terms of $\mathcal{N}=1$ superfields, one has two types of divergent diagrams, those of the chiral field propagator and of the gauge field one. The chiral vertices are finite due to the non-renormalization theorems [25] and for the gauge vertices one can choose the background gauge [26] where their divergent factors coincide with that of the gauge propagator. Thus, the only divergent structures are the field propagators only. Moreover, the gauge field propagator is not independent: its divergent structure is related to the chiral field propagators. This can be seen, for example, from the explicit form of the NSVZ gauge beta function [27] expressed in terms of the chiral field anomalous dimensions γ by

$$\beta_g = g^2 \frac{\sum T(R) - 3C(G) - \sum T(R)\gamma(R)}{1-2gC(G)}, \qquad g \equiv g^2/16\pi^2. \tag{2.3}$$

where $T(R)$ is the Dynkin index of a given representation R and $C(G)$ is the quadratic Casimir operator of the $\mathrm{SU}(N)$ gauge group. In the Leigh-Strassler deformed $\mathcal{N}=4$ SYM case one has the same field content as in $\mathcal{N}=4$ SYM, so $\sum T(R)=3C(G)$ and everything is defined by the chiral field anomalous dimension γ. Since conformal invariance is understood as the vanishing of the beta function, the Leigh-Strassler deformed theory is (super)conformal invariant on the sub-manifold in the coupling constant space which is defined by the following condition

$$\gamma(g,\ \{h_i\}) = 0, \tag{2.4}$$

where $\{h_i\} = (h_1,\ h_2,\ h_3)$. One can solve this condition (2.4) choosing the Yukawa couplings in the form of perturbation series over g [28]:

$$h_i = \alpha_{0i}g + \alpha_{1i}g^3 + \alpha_{2i}g^5 + \dots, i = 1\dots 3. \tag{2.5}$$

If the anomalous dimensions of the chiral fields vanish, so do the gauge and Yukawa beta functions and the theory is conformally invariant.

Conformal invariance also means that the theory is finite, i.e., all UV divergencies cancel (or in some gauges the sum of divergencies) and the renormalization factors Z (or their products) are equal to 1 or finite. In the context of dimensional regularization [29] this can be achieved by adding to expansion over g (2.5) a similar expansion over the parameter of dimensional regularization $\varepsilon = 4-D$, i.e., one has the two-fold expansion instead of one-fold one [30]

$$\begin{aligned} h_i = g & \left(a_i + \alpha_{0i}^{(1)}\varepsilon + \alpha_{0i}^{(2)}\varepsilon^2 + \dots + \alpha_{0i}^{(n-2)}\varepsilon^{n-2} + \alpha_{0i}^{(n-1)}\varepsilon^{n-1} + \alpha_{0i}^{(n)}\varepsilon^{n} + \dots\right) \\ & + g^3\left(\alpha_{1i}^{(0)} + \alpha_{1i}^{(1)}\varepsilon + \alpha_{1i}^{(2)}\varepsilon^2 + \dots + \alpha_{1i}^{(n-2)}\varepsilon^{n-2} + \alpha_{1i}^{(n-1)}\varepsilon^{n-1} + \dots\right) \\ & + g^5\left(\alpha_{2i}^{(0)} + \alpha_{2i}^{(1)}\varepsilon + \alpha_{2i}^{(2)}\varepsilon^2 + \dots + \alpha_{2i}^{(n-2)}\varepsilon^{n-2} + \dots\right) \\ & + \dots\dots\dots\dots\dots \end{aligned}$$

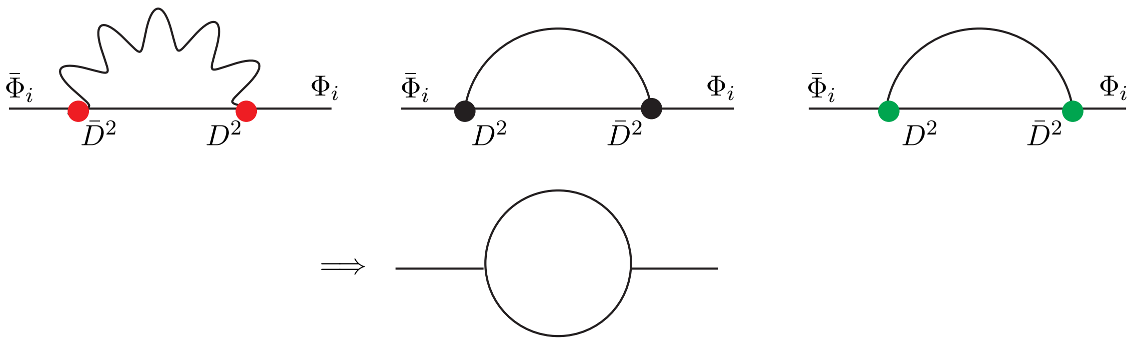

Figure 1: Supergraphs contributing to the chiral propagator at 1 loop and their scalar counterpart.

$$
\begin{aligned}
&+g^{2n-1}\left(\alpha^{(0)}_{n-2i}+\alpha^{(1)}_{n-2i}\varepsilon+\ldots\ldots\right)\\
&+g^{2n+1}\left(\alpha^{(0)}_{n-1i}+\ldots\right).
\end{aligned} \tag{2.6}
$$

In a given order of PT equal to n one needs all terms of the double expansion with a total power of $g^2 \cdot \varepsilon$ equal n. The existing freedom of choice of the coefficients $\alpha^{(m)}_{ki}$ is sufficient to get *simultaneously* the vanishing of the anomalous dimensions (read *conformal invariance*) and the pole terms in Z factors (read *finiteness*). The coefficients from $\alpha^{(0)}_{ni}$ to $\alpha^{(n)}_{0i}$ calculated in the n-th order of PT are related. One cannot put either of them to zero in an arbitrary way. For a more complete discussion and some examples of how these procedure works see our previous paper [18].

Our goal now is to calculate several terms of the double expansion (2.6) and to look for particular solutions when expansion breaks down at the first terms. In the case of a β-deformed SYM theory such a solution was found in [15] and corresponds to the real deformations, i.e., to $|q| = 1$.

In dimensional regularization (reduction) and $\overline{MS}$ renormalization scheme the anomalous dimension of a chiral superfield has the following form in the n-th order of PT:

$$
\gamma(g,\ \{h_i\}) = \sum_{k=1}^{n} k\ c_{1k}(g,\ \{h_i\}), \tag{2.7}
$$

where c_{1k} are the coefficients at the lowest order pole in Z_2^{-1}. In the 1-loop order one has for the chiral field renormalization constant

$$
Z_2^{-1} = 1 - \frac{N}{(4\pi)^2}\left(f(\{h_i\},N) - 2g^2\right)\frac{1}{\varepsilon}. \tag{2.8}
$$

Contributions to Z_2^{-1} are presented in figure 1 where red, black, and green dots correspond to chiral-gauge $\bar{\Phi}V\Phi$, chiral h_1, h_2 and chiral h_3 vertices. After performing D-algebra all diagrams in figure 1 reduce to the same scalar logarithmically divergent integral with different colour factors (hereafter we used SusyMath ver. 1.1 [31] and FeynCalc 5.1 [32] *Mathematica* packages to verify our calculations). From (2.8) one can see that

$$
c_{11} = -\frac{N}{(4\pi)^2}\left(f(\{h_i\},N) - 2g^2\right), \tag{2.9}
$$

where

$$f(\{h_i\},N) = \sum_{i,k=1}^{3} f_{ik}h_i\bar{h}_k = \left(1-\frac{2}{N^2}\right)(|h_1|^2+|h_2|^2)+\frac{2}{N^2}(h_1\bar{h}_2+h_2\bar{h}_1)+\left(1-\frac{4}{N^2}\right)|h_3|^2, \tag{2.10}$$

so the nonzero coefficients f_{ik} are

$$f_{11} = f_{22} = \left(1-\frac{2}{N^2}\right),\ f_{33} = \left(1-\frac{4}{N^2}\right),\ f_{21} = f_{12} = \frac{2}{N^2}, \tag{2.11}$$

where N is the number of colors of the gauge group SU(N).

Thus, the one-loop conformal condition takes the form

$$f(\{h_i\},N) - 2g^2 = 0. \tag{2.12}$$

To fulfil it, the coefficients in the expansion (2.6) $\{a_i\} = (a_1,\ a_2,\ a_3)$ must then satisfy the following requirement:

$$\sum_{i,k=1}^{3} f_{ik}a_i\bar{a}_k = 2. \tag{2.13}$$

To find other terms of expansion (2.6), one has to calculate the pole coefficients c_{ik} of (2.7) at higher orders of PT. For simplicity, we consider everywhere only the difference between the Leigh-Strassler deformed and the undeformed $\mathcal{N}=4$ SYM theory since calculating the difference we skip the calculation of many diagrams with gauge lines inside the diagrams [16]. The resulting expressions have some common structure in all orders of PT which simplifies the analysis:

Up to three loops in the planar case (or up to two loops in the non-planar case) the coefficients c_{ik} have the following form:

$$c_{nk} = (f(\{h_i\},N) - 2g^2)P_{nk}(h_i,g^2,N),\ n=1,\ldots,3,\ k=1,\ldots,n\,, \tag{2.14}$$

where $P_{nk}(h_i,g^2,N)$ is a homogenous polynomial of the form:

$$P_{nk}(\{h_i\},g^2,N) = \sum_{L=0}^{n-1}\sum_{i,k=1}^{3}(P_{nk})_{ikL}(h_i\bar{h}_k)^L(g^2)^{(n-1)-L},\ k=1,\ldots,n, \tag{2.15}$$

where $(P_{nk})_{ikL}$ are some real numbers. One can see that the one-loop conformal condition (2.12) is exact up to 3 loops in the planar case and up to two loops in the non-planar case. In higher orders new contributions appear and eq. (2.14) is modified.

2.1 Three-loop (non-planar limit) conformal condition

Starting from three loops in the nonplanar case one has the new contribution coming from the set of supergraphs with the "cross" topology (see figure 2). Equation(2.14) then takes the form

$$c_{nk} = (f(\{h_i\},N) - 2g^2)P_{nk}(\{h_i\},g^2,N) + G_{nk}(\{h_i\},N),\ n\geq 3,\ k=1,\ldots,n\,, \tag{2.16}$$

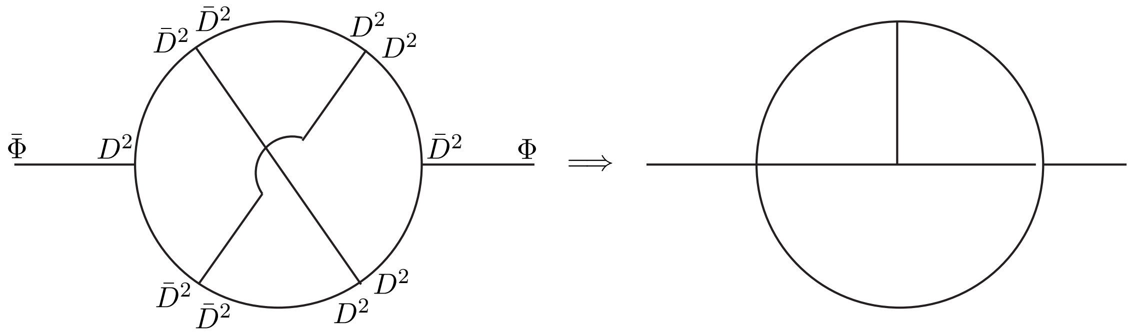

Figure 2: The topology of the relevant divergent non-planar supergraphs and their scalar counterpart at 3 loops

where

$$G_{nk}(\{h_i\}, N) = \sum_{i,p=1}^{3} (G_{nk})_{ip} (h_i \bar{h}_p)^n, \tag{2.17}$$

is a homogeneous polynomial, and

$$G_{nk}(\{a_i g\}, N) \neq 0, \tag{2.18}$$

i.e., G_{nk} do not vanish when applying the one loop conformal condition (2.12), and to achieve conformal invariance one has to take more terms of the double expansion (2.6). At this order of PT, to get simultaneously conformal and finite theory, one needs the following terms of expansion (2.6):

$$h_i = g\left(a_1 + \alpha_{0i}^{(2)} \varepsilon^2 + g^2 \alpha_{2i}^{(1)} \varepsilon^1 + g^4 \alpha_{4i}^{(0)}\right), \; i = 1, 2, 3. \tag{2.19}$$

The only nonvanishing contribution at this order of PT is G_{31}. The explicit form of G_{31} comes from the set of three loop nonplanar supergraphs with "cross" topology (figure 2). The D-algebra for every supergraph in this set is identical and leads to the same bosonic integral. It is easy to see that every supergraph with "cross" topology has no divergent subgraphs and every such supergraph contributes only to the simple pole coefficient in the singular part of the bare chiral propagator $\langle \Phi \bar{\Phi} \rangle_B$. So $G_{31} = -D_{31}$, where D_{31} is the pole coefficient in the singular part of the $\langle \Phi \bar{\Phi} \rangle_B$, and looks like

$$\begin{aligned} G_{31}(\{h_i\}, N) = & -\frac{1}{128} \frac{6\zeta(3)}{(4\pi)^6} \frac{N^2-4}{N^3} \times \\ & \Big\{ |h_1 - h_2|^2 \left(N^2 |h_1^2 + h_2^2 + h_1 h_2|^2 - 9N^2 |h_1|^2 |h_2|^2 + 5|h_1 - h_2|^4 \right) \\ & -18|h_3|^2 ((N^2-5)|h_1^2 + h_2^2|^2 - (N^2-10)(\bar{h}_1 \bar{h}_2 (h_2^2 + h_1^2) + c.c.) - 20|h_1|^2|h_2|^2) \\ & + \left(\bar{h}_3^3 (h_1 - h_2)((N^2+20)(h_1^2 + h_2^2) + 10(N^2-4) h_1 h_2) + c.c. \right) \\ & -8(N^2-10)(|h_3|^2)^3 \Big\}. \end{aligned} \tag{2.20}$$

Now we follow the standard procedure [18]: from the requirement of vanishing of the anomalous dimension one has up to 3 loops:

$$\gamma = c_{11} + 2c_{21} + 3c_{31} = 0 \tag{2.21}$$

and substituting (2.19) in (2.21) one has

$$1\ loop: \quad \sum_{i,k=1}^{3} f_{ik} a_i \bar{a}_k = 2, \tag{2.22}$$

$$3\ loops: \quad d_1 \sum_{i,k=1}^{3} f_{ik}(a_i \bar{\alpha}_{4k}^{(0)} + \alpha_{4i}^{(0)} \bar{a}_k) = -3 d_2 G_{31}^{\Sigma},$$

where hereafter we define

$$G_{31}(\{a_i g\}, N) = d_2 G_{31}^{\Sigma} g^6. \tag{2.23}$$

and $d_1 = \frac{N}{(4\pi)^2}$, $d_2 = -\frac{N^3}{128}\frac{6\zeta(3)}{(4\pi)^6}$. The explicit form of G_{31}^{Σ} is:

$$\begin{aligned} G_{31}^{\Sigma} = & \frac{N^2-4}{N^6}\{|a_1 - a_2|^2 \left(N^2|a_1^2 + a_2^2 + a_1 a_2|^2 - 9N^2|a_1|^2|a_2|^2 + 5|a_1 - a_2|^4\right) \\ & -18|a_3|^2 \left((N^2-5)|a_1^2 + a_2^2|^2 - (N^2-10)\left(\bar{a}_1\bar{a}_2(a_2^2 + a_1^2) + c.c.\right) - 20|a_1|^2|a_2|^2\right) \\ & + \left(\bar{a}_3^3(a_1 - a_2)((N^2+20)(a_1^2 + a_2^2) + 10(N^2-4)a_1 a_2) + c.c.\right) \\ & -8(N^2-10)(|a_3|^2)^3\}. \end{aligned} \tag{2.24}$$

To get $\alpha_{0i}^{(2)}$, according to [18], one has to consider $\langle \Phi\bar{\Phi}\rangle_B$. From the requirement of vanishing of all poles in $\langle \Phi\bar{\Phi}\rangle_B$ one has

$$6 d_1^3 \sum_{i,k=1}^{3} f_{ik}(a_i \bar{\alpha}_{0k}^{(2)} + \alpha_{0i}^{(2)} \bar{a}_k) - d_2 G_{31}^{\Sigma} = 0. \tag{2.25}$$

We used the RG equations to restore the necessary higher pole coefficients. To reach the total finiteness, one can use the remaining coefficients. From the requirement that $Z_2^{-1} = 1$ in 3 loops one gets, as in [18],

$$\begin{aligned} 3 d_1^2 \sum_{i,k=1}^{3} f_{ik}(a_i \bar{\alpha}_{2k}^{(1)} + \alpha_{2i}^{(1)} \bar{a}_k) + d_1 \sum_{i,k=1}^{3} f_{ik}(a_i \bar{\alpha}_{4k}^{(0)} + \alpha_{4i}^{(0)} \bar{a}_k) & \\ + 6 d_1^3 \sum_{i,k=1}^{3} f_{ik}(a_i \bar{\alpha}_{0k}^{(2)} + \alpha_{0i}^{(2)} \bar{a}_k) g^6 + d_2 G_{31}^{\Sigma} & = 0, \end{aligned} \tag{2.26}$$

or using (2.22), (2.25):

$$3 d_1^2 \sum_{i,k=1}^{3} f_{ik}(a_i \bar{\alpha}_{2k}^{(1)} + \alpha_{2i}^{(1)} \bar{a}_k) - d_2 G_{31}^{\Sigma} = 0. \tag{2.27}$$

Putting all together we obtain that up to 3 loops $\{h_i\}$ must satisfy the following condition:

$$\begin{aligned} \sum_{i,k=1}^{3} f_{ik} h_i \bar{h}_k & = \left(1 - \frac{2}{N^2}\right)(|h_1|^2 + |h_2|^2) + \frac{2}{N^2}(h_1\bar{h}_2 + h_2\bar{h}_1) + \left(1 - \frac{4}{N^2}\right)|h_3|^2 \\ & = g^2 \left\{2 - \frac{\zeta_3}{128} G_{31}^{\Sigma}\varepsilon^2 - \frac{2\zeta_3}{128} G_{31}^{\Sigma}\left(\frac{g^2 N}{16\pi^2}\right)\varepsilon + \frac{18\zeta_3}{128} G_{31}^{\Sigma}\left(\frac{g^2 N}{16\pi^2}\right)^2\right\} \end{aligned} \tag{2.28}$$

For the bare couplings one has:

$$\sum_{i,k=1}^{3} f_{ik}(h_i\bar{h}_k)|_B = g_B^2\left\{2-\frac{\zeta_3}{128}G_{31}^{\Sigma}\varepsilon^2+\dots\right\} \tag{2.29}$$

For any values of the coefficients in (2.19) which satisfy (2.28) the theory is conformally invariant and finite up to three loops. In the planar limit we see from(2.24) that the coefficient G_{31}^{Σ} vanishes, which leads us to the one-loop conformal condition.

2.2 Four-loop (planar limit) conformal condition

The situation is simplified in the planar (large N of the SU(N) gauge group) limit. In this case, in the one-loop conformal condition (2.10) only the diagonal terms $f_{ik},\ i=k$ survive

$$f(\{h_i\},N) = \sum_{i,k=1}^{3} f_{ik}h_i\bar{h}_k = |h_1|^2+|h_2|^2+|h_3|^2, \tag{2.30}$$

so from (2.12) one has

$$|h_1|^2+|h_2|^2+|h_3|^2-2g^2=0. \tag{2.31}$$

At three loops all $G_{ik}=0$ (note that the set of supergraphs with "cross" topology does not survive in the planar limit). At four loops the only nonvanishing contribution to G_{41} comes from the set of planar supergraphs with the new "ladder" topology (see figure 3). The D-algebra for every supergraph in this set is identical and leads to the same bosonic integral. It is easy to see that every chiral supergraph with the "ladder" topology has no divergent subgraphs. The contribution of this set of chiral supergraphs to the chiral propagator renormalization constant in the planar limit is:

$$\begin{aligned} c_{41}(\{h_i\},g^2,N) = {} & \frac{5}{2}\zeta(5)\frac{N^4}{(4\pi)^8}\{(|h_1|^2+|h_2|^2+|h_3|^2)^4-(2g^2)^4+(|h_1|^2-|h_2|^2)^4+(|h_3|^2)^4 \\ & +6(|h_3|^2)^2(|h_1|^2+|h_2|^2)^2+24|h_3|^2|h_1|^2|h_2|^2(|h_1|^2+|h_2|^2)+ \\ & +8h_3^3(|h_2|^2\bar{h}_1^3-|h_1|^2\bar{h}_2^3)+8\bar{h}_3^3(|h_2|^2h_1^3-|h_1|^2h_2^3) \\ & -8|h_3|^2(h_2^3\bar{h}_1^3+h_1^3\bar{h}_2^3)-4|h_3|^2(|h_1|^2+|h_2|^2)^3-4(|h_3|^2)^3(|h_1|^2+|h_2|^2)\}. \end{aligned}$$

Hereafter the chiral-gauge $\bar{\Phi}V\Phi$ contributions proportional to $|h_1|^2+|h_2|^2+|h_3|^2-2g^2$ are omitted. Note that in this case $G_{41}=c_{41}$ and does not vanish at the one-loop conformal condition.

With account of nonvanishing contribution to G_{41} one needs the following terms of expansion (2.6):

$$h_i = g\left(a_i+\alpha_{0i}^{(3)}\varepsilon^3+g^2\alpha_{2i}^{(2)}\varepsilon^2+g^4\alpha_{4i}^{(1)}\varepsilon+g^6\alpha_{6i}^{(0)}\right),\ i=1,2,3. \tag{2.32}$$

From the requirement of vanishing of the anomalous dimension $\gamma = c_{11}+2c_{21}+3c_{31}+4c_{41}=0$, one finds

$$1\ loop:\ \sum_{i,k=1}^{3} f_{ik}a_i\bar{a}_k = 2, \tag{2.33}$$

$$4\ loops:\ d_1[(\overline{a}_1\alpha_{31}^{(0)}+a_1\overline{\alpha}_{31}^{(0)})+(\overline{a}_2\alpha_{32}^{(0)}+a_2\overline{\alpha}_{32}^{(0)})+(\overline{a}_3\alpha_{33}^{(0)}+a_3\overline{\alpha}_{33}^{(0)})]=-4d_2G_{41}^{\Sigma},$$

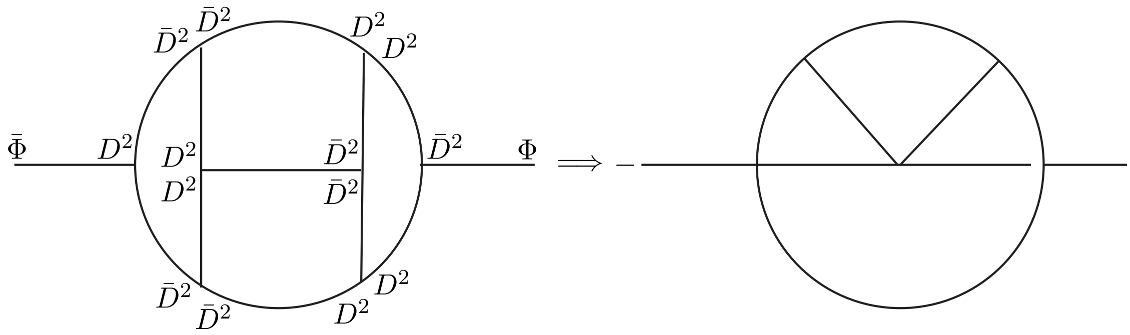

Figure 3: The topology of the relevant divergent planar supergraphs and their scalar counterpart at 4 loops

where now $d_1 = \frac{N}{(4\pi)^2}$, $d_2 = \frac{5}{2}\frac{\zeta(5)N^4}{(4\pi)^8}$. The explicit form of G_{41}^{Σ} is:

$$\begin{aligned} G_{41}^{\Sigma} = \Big\{ & (a_3\overline{a}_3)^4 + (a_1\overline{a}_1 - a_2\overline{a}_2)^4 + 6(a_3\overline{a}_3)^2(a_1\overline{a}_1 + a_2\overline{a}_2)^2 \\ & +24a_1\overline{a}_1a_2\overline{a}_2a_3\overline{a}_3(a_1\overline{a}_1 + a_2\overline{a}_2) + 8a_3^3(a_2\overline{a}_2\overline{a}_1^3 - a_1\overline{a}_1\overline{a}_2^3) \\ & +8\overline{a}_3^3(a_2\overline{a}_2a_1^3 - a_1\overline{a}_1a_2^3) - 8a_3\overline{a}_3(\overline{a}_1^3a_2^3 + \overline{a}_2^3a_1^3) \\ & -4ca_3\overline{a}_3(a_1\overline{a}_1 + a_2\overline{a}_2)^3 - 4(a_3\overline{a}_3)^3(a_1\overline{a}_1 + a_2\overline{a}_2)\Big\}. \end{aligned} \tag{2.34}$$

To get $\alpha_{0i}^{(3)}$, according to [18], one has to consider the bare propagator. Since the only nontrivial graph giving contribution to G_{41} has no divergent subgraphs, the essential singular part of the bare propagator is $D_{41} = -c_{41}$. From the requirement of vanishing of all poles in $\langle\Phi\bar{\Phi}\rangle_B$ one has

$$\widehat{P}_{44}g^2\left((\overline{a}\alpha_{01}^{(3)} + a\overline{\alpha}_{01}^{(3)}) + (\overline{a}_2\alpha_{02}^{(3)} + a_2\overline{\alpha}_{02}^{(3)}) + (\overline{a}_3\alpha_{03}^{(3)} + a_3\overline{\alpha}_{03}^{(3)})\right) - d_2\widehat{G}_{41} = 0. \tag{2.35}$$

After calculating the value of $\widehat{P}_{44}$ from the pole equations we find

$$d_1^4[(\overline{a}\alpha_{01}^{(3)} + a\overline{\alpha}_{01}^{(3)}) + (\overline{a}_2\alpha_{02}^{(3)} + a_2\overline{\alpha}_{02}^{(3)}) + (\overline{a}_3\alpha_{03}^{(3)} + a_3\overline{\alpha}_{03}^{(3)})] = d_2\frac{G_{41}^{\Sigma}}{9}. \tag{2.36}$$

To reach total finiteness, one can use the remaining coefficients. From the requirement that $Z_2^{-1} = 1$ in four loops one gets, as in [18],

$$\begin{aligned} d_1^3[(\overline{a}_1\alpha_{21}^{(2)} + a_1\overline{\alpha}_{21}^{(2)}) + (\overline{a}_2\alpha_{22}^{(2)} + a_2\overline{\alpha}_{22}^{(2)}) + (\overline{a}_3\alpha_{23}^{(2)} + a_3\overline{\alpha}_{23}^{(2)})] &= -\frac{2d_2}{3}G_{41}^{\Sigma}, \\ d_1^2[(\overline{a}_1\alpha_{41}^{(1)} + a_1\overline{\alpha}_{41}^{(1)}) + (\overline{a}_2\alpha_{42}^{(1)} + a_2\overline{\alpha}_{42}^{(1)}) + (\overline{a}_3\alpha_{43}^{(1)} + a_3\overline{\alpha}_{43}^{(1)})] &= 2d_2G_{41}^{\Sigma}. \end{aligned} \tag{2.37}$$

Again we have the finite and conformal theory up to four loops if the renormalized Yukawa couplings are chosen to satisfy the condition

$$\begin{aligned} \sum_{i,k=1}^{3} f_{ik}h_i\bar{h}_k = |h_1|^2 + |h_2|^2 + |h_3|^2 = g^2\Big\{ & 2 + \frac{5}{18}\zeta_5G_{41}^{\Sigma}\varepsilon^3 + \frac{5}{3}\zeta_5G_{41}^{\Sigma}\left(\frac{g^2N}{16\pi^2}\right)\varepsilon^2 \\ & +5\zeta_5G_{41}^{\Sigma}\left(\frac{g^2N}{16\pi^2}\right)^2\varepsilon + 10\zeta_5G_{41}^{\Sigma}\left(\frac{g^2N}{16\pi^2}\right)^3 + \dots\Big\}, \end{aligned} \tag{2.38}$$

where G^{Σ}_{41} was given above (2.34). For the bare couplings one has

$$|h_1|^2_B + |h_2|^2_B + |h_3|^2_B = g^2_B \left\{ 2 + \frac{5}{18}\zeta_5 G^{\Sigma}_{41}\varepsilon^3 + \dots \right\}. \tag{2.39}$$

This again permits, in particular, the value of $|q| \neq 1$, thus allowing one to obtain a complex deformation of the $\mathcal{N} = 4$ SYM theory with arbitrary complex β.

3. Unitarity transformation

As was first noticed in [33], considering the full Leigh-Strassler deformation one can find special points in the parameter space of $\{h_1, h_2, h_3\}$ at which the theory is unitary equivalent to the β-deformed $\mathcal{N} = 4$ SYM theory.

Consider a general unitary matrix U(3) ($UU^+ = 1$). It depends on 9 parameters. Three of them are the Euler angles and the other six are the phases. Similarly to the quark mixing, five of six phases can be eliminated by the redefinition of the chiral superfields. What is left has the standard Cabbibo-Kobayashi-Maskawa form [34]

$$U = \begin{pmatrix} c_1 & c_3 s_1 & s_1 s_3 \\ -c_2 s_1 & c_1 c_3 - e^{iy} s_2 s_3 & e^{iy} c_3 s_2 + c_1 c_2 s_3 \\ s_1 s_2 & -c_1 c_3 s_2 - e^{iy} c_2 s_3 & e^{iy} c_2 c_3 - c_1 s_2 s_3 \end{pmatrix}$$

where $s_i = \sin(x_i)$ and $c_i = \cos(x_i)$.

We take now the β-deformed theory and make an arbitrary unitary transformation of the fields

$$\Phi_i \;=\; U_{ij}\Psi_j. \tag{3.1}$$

After that we demand the new theory to be of the Leigh-Strassler type. It means the absence of nondiagonal terms like $Tr(\Psi_i\Psi_j\Psi_j)$ $i \neq j$. In the above-defined parametrization of the unitary matrix this procedure leads to the **full** Leigh-Strassler deformation theory provided the parameters take on the following values:

$$\begin{cases} x_1 = \pm \arccos(\frac{1}{\sqrt{3}}) + \pi k, \\ x_2 = \frac{\pi}{4} + \frac{\pi l}{2}, \\ x_3 = \frac{\pi}{4} + \frac{\pi m}{2}, \\ y = \frac{\pi}{2} + \pi n. \end{cases} \tag{3.2}$$

It should be mentioned that besides the absence of the mixed terms we would like also to get the coefficients of $Tr(\Psi_i^3)$ to be equal. Indeed, the absence of non-diagonal terms in our case automatically leads to the equal coefficients of $Tr(\Psi_i^3)$ up to the phases $e^{i\alpha_i}$. However, these phases can be eliminated by the additional phase rotation of the chiral superfields $\Psi_i \to e^{-i\frac{\alpha_i}{3}}\tilde{\Psi}_i$ and one gets the theory of exactly the Leigh-Strassler form.

As the result, the superpotential which is obtained from the β-deformed SYM theory by unitary transformation (3.1) with parameters fixed by (3.2) has the form

$$W = iTr\left(\tilde{h_1}\Psi_1\Psi_2\Psi_3 - \tilde{h_2}\Psi_1\Psi_3\Psi_2\right) + i\frac{\tilde{h_3}}{3}\sum_{i=1}^{3} Tr(\Psi_i^3), \tag{3.3}$$

where

$$\begin{cases} \tilde{h_1} = i(a-b) \\ \tilde{h_2} = i(a+b) \\ \tilde{h_3} = 2ib \end{cases} \text{or} \begin{cases} \tilde{h_1} = e^{\pm\frac{\pi}{3}}(a-b) \\ \tilde{h_2} = e^{\pm\frac{\pi}{3}}(a+b) \\ \tilde{h_3} = -2ib \end{cases} \tag{3.4}$$

Here the factor $e^{\pm\frac{\pi}{3}}$ has the origin from the different phases of the $Tr(\Psi_i^3)$ term for different i. The parameters a and b are linked with the original couplings h_1 and h_2 by

$$\begin{cases} a = \pm\frac{1}{2}(h_1 + h_2), \\ b = \pm\frac{1}{i2\sqrt{3}}(h_1 - h_2). \end{cases} \tag{3.5}$$

The signs in expressions for a and b can be chosen independently.

The chiral propagators calculated in the full Leigh-Strassler deformed theory (3.3) with the couplings chosen as (3.4), (3.5) will be the same as calculated in the β-deformed theory. This provides us with nontrivial check of the calculations made in the Leigh-Strassler deformed theory. Namely, taking expressions (2.20), (2.32) and after making a substitution (3.4), (3.5) one obtains the known results for the β-deformed theory [16, 15].

4. Exploring the conformal conditions

Let us consider the calculated expressions for G_{31} in the non-planar case and G_{41} in the planar case and try to find such values of (h_1, h_2, h_3) when these quantities vanish meaning that the one-loop conformal condition is valid up to three or four loops. Knowing that in the case of the real beta deformation in the planar limit the one-loop conformal condition is exact we are interested in finding new solutions in the full Leigh-Strassler deformed theory for which the one-loop conformal condition is also exact.

First of all, similarly to the β-deformed theory, we have not found any solution for vanishing of G_{31} in the nonplanar case which has a simple form and might be valid in any order of PT.

In the planar case, on the contrary, we found two families of simple solutions of the equation $G_{41} \ = \ 0$.

Solution # 1:

$$\begin{cases} \tilde{h_1} = ge^{i\alpha}(A-B), \\ \tilde{h_2} = ge^{i\alpha}(A+B), \\ \tilde{h_3} = 2ge^{i\alpha}B, \end{cases} \tag{4.1}$$

where A, B, α are arbitrary *real* numbers. The one-loop conformal condition brings us to the following relation between A and B:

$$B^2 = \frac{1-A^2}{3}.$$

If this condition is satisfied, then $G_{41} = 0$ for arbitrary α and $-1 \leq A \leq 1$.

However, it is easy to see that solution # 1 coincides with the left part of (3.4). This means that the obtained theory is unitary equivalent to the β-deformed case and is exactly conformal in the planar limit.

Solution # 2:

$$\begin{cases} \tilde{h_1} = -ge^{i\alpha}, \\ \tilde{h_2} = 0, \\ \tilde{h_3} = ge^{i\beta}, \\ \alpha - \beta \neq \frac{2\pi m}{3} \end{cases} \quad \text{or} \quad \begin{cases} \tilde{h_1} = 0, \\ \tilde{h_2} = ge^{i\alpha}0, \\ \tilde{h_3} = ge^{i\beta}, \\ \alpha - \beta \neq \frac{2\pi m}{3} \end{cases} \tag{4.2}$$

These two cases are equivalent. For $\alpha - \beta = \frac{2\pi m}{3}$ the obtained theory is unitary equivalent to the real β-deformed one, but for arbitrary real values of α and β this is genuine.

Solution # 3:

$$\begin{cases} \tilde{h_1} = g(A - iB), \\ \tilde{h_2} = g(A + iB), \\ \tilde{h_3} = -4igB, \end{cases} \tag{4.3}$$

where A and B are equal to $A = \pm\frac{1}{2}, B = \pm\frac{1}{2\sqrt{3}}$. This solution is unitary equivalent to solution # 2.

Thus, the only nontrivial solution that exists in the planar limit and leads to conformal theory (up to 4 loops at least) corresponds to the superpotential which can be written in the form

$$\mathcal{W} = ih \int d^6z \left(qTr\Phi_1\Phi_2\Phi_3 - \frac{1}{q} \sum_{i=1}^{3} \frac{Tr(\Phi_i^3)}{3} \right). \tag{4.4}$$

where $|h|^2 = g^2$ and $|q| = 1$. The case $q = e^{i\frac{\pi n}{3}}$ brings us back to the real β-deformed theory. In the next section we consider some properties of this theory.

5. Exact conformal invariance?

One may wonder if the theory defined by the superpotential (4.4) is exactly conformal in the planar limit when $|q| = 1$ precisely like the β-deformed one. To understand whether the conformal condition is exhausted by one loop, we consider the corrections to the chiral propagator being interested in the phase-dependent ones. Due to the unitary equivalence to real β-deformed theory for particular values of the phase the absence of phase-dependent terms would mean the exact conformal invariance of the theory.

One can observe that the conformal condition in the planar limit is related to topology of the chiral diagrams [35]. The one-loop conformal condition stays valid in higher orders when the diagrams contain the "bubbles" on the lines. The next structure that might emerge is a triangle, but since the propagators are always chiral-antichiral such a kind of diagrams is forbidden. The next structure is the "box" present in the "ladder" type diagrams. It appears for the first time in four-loops and does not contain a phase factor in the planar limit. To get the phase factor, one should consider more complicated polygons.

From the superpotential (4.4) one can notice that only phase-dependent structures that can emerge are of the form

$$(|h_3|^2)^n(|h_1|^2)^l[(h_3\bar{h_1})^{3k} + (\bar{h}_3h_1)^{3k}], k = 0, 1, \dots.$$

Hence, if $h_1 = hq,\ h_3 = \frac{h}{q}$, $q = e^{i\gamma}$ the only phase-dependent contribution looks like

$$const \times \cos(6k\gamma).$$

Since we know that when $q = e^{i\frac{\pi n}{3}}$ the theory is unitary equivalent to the real β-deformed one, it should be exactly conformal for $\gamma = \pi n/3$. This corresponds to $\cos(6k\gamma) = \cos(2\pi kn) = 1$ for arbitrary k and n. Moreover, it is clear that the substitution

$$\gamma \to \gamma + \frac{\pi}{3}$$

does not change anything and if a theory is exactly conformal for some γ, it automatically conformal for

$$\gamma + \frac{\pi n}{3}.$$

This is similar to the beta deformed case where such an equivalence was of the form

$$\beta \to \beta + \pi n.$$

So the crucial question is whether it is possible to construct a diagram which is phase-dependent in the planar limit. This happened to be not a simple task for the following reasons:

1. All possible phase-dependent "boxes" are suppressed in the planar limit. Thus, the possible phase-dependent diagram should contain more complicated structures.

2. The diagram containing a polygon higher than the "box" in which all phase-dependence is encoded has many external legs. Hence, to reduce their number to two in order to get the chiral propagator and keeping only the planar diagrams, one has to make new "boxes" which again contain no phases. As the result, at least up to ten loops, one cannot construct a potentially phase-dependent diagram in the planar limit. We assume, though we have no rigorous proof yet, that in the planar limit such a phase-dependent structure does not emerge in any order of PT.

The extra argument for the exact conformal invariance of the presented theory comes from the the investigation of the integrability properties of the one-loop dilatation operator in the full Leigh-Strassler theory made in [36]. The above suggested solution corresponds to the points in the parameter space where the theory was found to be integrable in the planar limit. This seems to be similar to the β-deformed case where the exact conformal condition is accompanied with the integrability [37].

Thus our conjecture is that the theory defined by the superpotential (4.4) with $|q| = 1$ is exactly conformal in the planar limit.

6. Conclusion

We have investigated here the conformal conditions for the full Leigh-Strassler deformation of the $\mathcal{N} = 4$ SYM theory both in the planar and nonplanar cases. The conformal condition was found up to four loops in the planar limit and up to three loops in non-planar case. We would like to emphasize that the obtained theory is *simultaneously* conformal invariant and finite since these two requirements are *identical*. This can be achieved properly adjusting the Yukawa couplings order by order in PT. In the framework of dimensional regularization this requires the double series over the gauge coupling g and the parameter of dimensional regularization ε.

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Since in the full Leigh-Strassler deformation of the $\mathcal{N} = 4$ SYM theory there is an extra coupling constant, we have more freedom in our theory. Thus we looked for the solutions where the one-loop conformal condition is exact and at the same time which are not obtainable from the real beta deformation of the $\mathcal{N} = 4$ SYM theory by unitary transformation. We did not find such solutions in the nonplanar case but in the planar limit we found one potentially interesting solution. We made certain that in the planar limit the one-loop conformal condition in this case is valid up to ten loops and we present the arguments that it might also be valid in any order of PT.

If our conjecture is true, then it will be interesting to understand the nature of this exact conformal condition from the field theory side as well as from the point of view of the dual gravity background. While constructing gravity dual background for the β-deformed theory the important role was played by the global $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry of the Lagrangian. The theory presented here has no continuous global symmetries but at some points of the parameter space it is unitary equivalent to the β-deformed theory. This suggests some common features hidden so far. From this point of view constructing the dual description would be very interesting.

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